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Representation of Entire Harmonic Functions by Given Values

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1. INTRODUCTION AND STATEMENT OF RESULTS

In [3], Boas proved the following uniqueness theorem for harmonic functions:

Let u be a real-valued entire harmonic function of exponential type less than π such that $u(n) = u(n+i) = 0$ for all integers n, where $i^2 = -1$. Then u is identically zero.

He also asked some questions including the following two:

(1) Is it possible to reconstruct an entire harmonic function u of exponential type less than π from its values on the lattice points n, $n+i$ $(n = 0, \pm 1, \pm 2, \ldots)$?

(2) If an entire function of exponential type $\tau < \pi$ is bounded at all the integers, then, by a theorem of Cartwright [2, Theorem 10.2.3], it is bounded on the whole real line. Does a corresponding statement hold for a real-valued entire harmonic function of exponential type less than π which is bounded on the lattice points n, $n+i$ $(n = 0, \pm 1, \pm 2, \ldots)$?

As regards the first question Ching and Chui gave an affirmative answer under the hypothesis that $u(x)$ and $u(x+i)$ belong to L^2 on the real line; they indeed found an explicit interpolation formula which represents u by its values on the lattice points n, $n+i$ $(n = 0, \pm 1, \pm 2, \ldots)$. Andersen [1] showed that the formula of Ching and Chui also holds under somewhat

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weaker assumptions, which are satisfied if, for example, $u(x)$ and $u(x+i)$ belong to L^p $(0<p<\infty)$, but do not cover the case L^∞. We will show that Question 1 of Boas has an affirmtive answer under our considerably weaker condition (4).

Concerning the second question it was observed by A. M. Trembinska that Cartwright's theorem extends directly to harmonic functions, i.e., if u is a real harmonic function of exponential type $<\pi$ with $|u(n)|<M$ for $n=0, \pm 1, \pm 2,\ldots$, then $|u(x)|<M_1$ for $x\in\mathbb{R}$. The simple example $u(z):=xy$ shows that a harmonic function of exponential type $<\pi$ may be bounded on the real axis without being bounded on any parallel line. However, we shall show that boundedness at the lattice points n, $n+i$ $(n=0, \pm 1, \pm 2,\ldots)$ implies boundedness in every strip of finite width parallel to the real axis.

Given $\alpha>1$ and $\delta>0$ we can construct [5, see Example] an entire function $\psi:=\psi(\alpha,\delta,\cdot)$ of exponential type at most δ such that

(i) $\psi(0)=1$,

(ii) $\psi(-x)=\psi(x)\in\mathbb{R}$ for $x\in\mathbb{R}$,

(iii) $|\psi(x)|=O(\exp(-|x|/(\log|x|)^\alpha)$ as $x\to\pm\infty$.

Now put

$$\Lambda(z):=\Lambda(\alpha,\delta,z):=\psi\left(\frac{1+\alpha}{2},\frac{\delta}{2},z\right)\frac{\sin\pi z}{\pi z}. \tag{1}$$

Then Λ is an entire function of exponential type less than $\pi+\delta$ satisfying (i)–(iii), and in addition

(iv) $\Lambda(n)=0$ for $n=\pm 1, \pm 2, \pm 3,\ldots$.

Because of (iii) the restriction of Λ to the real line is in $L^1\cap L^2$. Hence by the theory of Fourier transforms

$$\begin{aligned}\lambda(t):=\lambda(\alpha,\delta,t)&:=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ixt}\Lambda(x)\,dx\\ &=\frac{1}{2\pi}\int_{-\infty}^{\infty}\cos(xt)\,\Lambda(x)\,dx\end{aligned} \tag{2}$$

exists and belongs to $L^1\cap L^2$; by the Paley–Wiener theorem [2, p. 103] it must vanish for $|t|\geqslant\tau:=\pi+\delta$. Finally, for $z=x+iy$ $(x,y\in\mathbb{R})$ we define

$$H(z):=H(\alpha,\delta,z):=\int_{-\tau}^{\tau}\frac{\sinh(t(1-y))}{\sinh t}\cos(tx)\,\lambda(t)\,dt. \tag{3}$$

We are now ready to state our

THEOREM. *Let u be a real-valued entire harmonic function of exponential type $\sigma < \pi$ such that for some $\gamma > 1$ and all integers n*

$$\left.\begin{array}{l} |u(n)| \\ |u(n+i)| \end{array}\right\} = O(\exp(|n|/(\log|n|)^{\gamma}) \qquad as \quad n \to \pm\infty. \tag{4}$$

Then for $\alpha \in (1, \gamma)$ and $\delta \in (0, \pi - \sigma)$

$$u(z) = \sum_{n=-\infty}^{\infty} u(n)\, H(z-n) + u(n+i)\, H(\bar{z} - n + i). \tag{5}$$

The series converges absolutely and uniformly on every compact subset of $\mathbb{C}$.

Remark 1. The condition (4) can be somewhat relaxed. However, our method does require an asymptotic growth not exceeding $O(\exp(w(|n|)))$ on the right-hand side of (4), where the function w must be such that

$$\int_1^{\infty} \frac{|w(x)|}{x^2}\, dx < \infty$$

(see [5]).

Remark 2. The function H in (3) is not easy to handle. But once an efficient subroutine for $H(z)$ has been established, the representation formula (5) can be of numerical interest, even in the cases where the formula of Ching and Chui is applicable. Indeed, the infinite series converges quite rapidly (see (9) below) and a relatively small number of terms will suffice to guarantee a desired numerical accuracy.

From the theorem we shall deduce the following:

COROLLARY. *If a real-valued entire harmonic function u of exponential type $\sigma < \pi$ is bounded by K on the lattice points n, $n+i$ $(n = 0, \pm 1, \pm 2, \ldots,)$ then it is bounded by $K \cdot M$ in every strip $\{z : |\mathrm{Im}\, z| \leqslant y_0\}$, where M depends only on σ and y_0.*

Remark 3. Note that in the corollary the condition $\sigma < \pi$ cannot be relaxed. In fact, the entire harmonic function

$$u : x + iy \mapsto ce^{\pi y} \sin \pi x$$

being of exponential type π vanishes on the lattice points n, $n+i$, whereas at a point $\frac{1}{2} + iy$ it can be made as large as we want by choosing c appropriately.

2. Lemmas

Lemma 1. *If $\Lambda := \Lambda(\alpha, \delta, \cdot)$ is the function defined in* (1), *then for every* $y_0 > 0$

$$|\Lambda(x+iy)| = O(\exp(-|x|/(\log|x|)^{\alpha})) \qquad as \quad x \to \pm\infty \tag{6}$$

uniformly for $|y| \leqslant y_0$.

Proof. Obviously it is sufficient to prove (6) for $y \geqslant 0$ and $x \to +\infty$ only. Consider the function

$$g: z \mapsto \exp(-z/(\log z)^{\beta}), \qquad \beta = (1+\alpha)/2,$$

which is holomorphic in the half-plane $\{z : \operatorname{Re} z \geqslant 2\}$ and is different from zero. Consequently, the function

$$h: z \mapsto \frac{\Lambda(z+2)}{g(z+2)}$$

is holomorphic and of exponential type in the first quadrant. Moreover (iii) and (1) imply

$$\lim_{x \to +\infty} h(x) = 0.$$

Then, according to a well-known result [2, Theorem 6.2.8]

$$\lim_{x \to +\infty} h(x+iy) = 0$$

uniformly for $y \in [0, y_0]$. From this the result follows immediately.

Lemma 2. *Let f be an entire function of exponential type $\sigma < \pi$ such that for some $\gamma > 1$ and all integers n*

$$|f(n)| = O(\exp(|n|/(\log|n|)^{\gamma})) \qquad as \quad n \to \pm\infty.$$

Choose $\alpha \in (1, \gamma)$ and $\delta \in (0, \pi - \sigma)$. Then for the function $\Lambda := \Lambda(\alpha, \delta, \cdot)$ defined in (1)

$$f(z) = \sum_{n=-\infty}^{\infty} f(n)\, \Lambda(z-n), \tag{7}$$

where the series converges absolutely and uniformly on every compact subset of $\mathbb{C}$.

Proof. For $w \in \mathbb{C}$ consider

$$g_w: \quad z \mapsto f(z)\, \psi\left(\frac{1+\alpha}{2}, \frac{\delta}{2}, w - z\right).$$

This is an entire function of exponential type less than π such that

$$\sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left| \frac{g_w(n)}{n} \right| < \infty.$$

Hence [6, p. 184, Sect. 4.3.2]

$$g_w(z) = \sum_{n=-\infty}^{\infty} f(n)\, \psi\left(\frac{1+\alpha}{2}, \frac{\delta}{2}, w - n\right) \frac{\sin \pi(z-n)}{\pi(z-n)}.$$

Now, setting $w := z$ we obtain the desired representation. That the series converges absolutely and uniformly on all compact subsets of $\mathbb{C}$ is implied bu Lemma 1.

LEMMA 3. *The function H defined in* (3) *is an entire harmonic function of exponential type less than $\tau := \pi + \delta$ satisfying*

$$H(x) = \Lambda(x) \qquad \text{for} \quad x \in \mathbb{R}, \tag{8}$$

where Λ is given in (1),

$$H(x+i) = 0 \qquad \text{for} \quad x \in \mathbb{R},$$

and

$$|H(x+iy)| = O(\exp(-|x|/(\log|x|)^{\alpha})) \qquad \text{as} \quad x \to \pm\infty \tag{9}$$

uniformly in the strip $\{x+iy : |y| \leqslant y_0\}$ for every $y_0 > 0$.

Proof. Only (8) and (9) may not be obvious. For $z = x \in \mathbb{R}$ equality (3) reduces to

$$H(x) = \int_{-\tau}^{\tau} \cos(tx)\, \lambda(t)\, dt = \int_{-\tau}^{\tau} e^{itx} \lambda(t)\, dt.$$

Comparing with (2), the well-known relationship between the Fourier transform and its inverse says that $H(x) \equiv \Lambda(x)$ on the real line.

Since by the Riemann–Lebesgue lemma (see, e.g., [7, p. 46, Theorem 4.6]) for every fixed $y \in \mathbb{R}$

$$\lim_{x \to \pm\infty} H(x+iy) = 0 \tag{10}$$

we see after a short reflection that in the strip

$$S := \{z \in \mathbb{C} : 0 \leqslant \operatorname{Im} z \leqslant 1\}$$

the harmonic function H can be reconstructed from its values on the boundary using Poisson's formula transformed to S. This way we obtain for $y \in (0, 1]$

$$H(x+iy) = \frac{1}{2} \int_{-\infty}^{\infty} A(x+t) \frac{\sin \pi y}{\cosh \pi t - \cos \pi y}\, dt.$$

As $H(x-iy) \equiv H(-x+iy)$ we may restrict ourselves to the case $x > 1$. Now, splitting the integral into

$$\int_{-\infty}^{\infty} \cdots = \int_{-\infty}^{-x \log x} \cdots + \int_{-x \log x}^{x \log x} \cdots + \int_{x \log x}^{+\infty} \cdots$$

and taking into account that

$$\int_{-\infty}^{\infty} \frac{\sin \pi y}{\cosh \pi t - \cos \pi y}\, dt = 2(1-y)$$

we readily see that (9) holds uniformly on S.

Finally, let F be an entire function such that $H(z) = \operatorname{Re} F(z)$ and consider

$$G(z) := \tfrac{1}{2}(F(z) + \overline{F(\bar{z})}).$$

Then, clearly

$$G(x) = H(x) = A(x) \qquad \text{for all} \quad x \in \mathbb{R}$$

which implies

$$G(z) = A(z) \qquad \text{for all} \quad z \in \mathbb{C}.$$

Hence

$$\operatorname{Re} A(z) = \tfrac{1}{2}(H(z) + H(\bar{z})). \tag{11}$$

Since

$$H(x+i) = 0 \qquad \text{for all} \quad x \in \mathbb{R},$$

we conclude with the help of the Schwarz reflection principle applied to iF with respect to the line $i + \mathbb{R}$ that

$$H(z+i) = -H(\bar{z}+i). \tag{12}$$

From (11) and (12) we deduce

$$H(z+2i) = H(z) - 2 \operatorname{Re} A(z) \qquad \text{for all} \quad z \in \mathbb{C}. \tag{13}$$

Now (12), (13), and Lemma 1 enable us to extend the validity of (9) from S to any other strip with boundaries parallel to the real line. This completes the proof of Lemma 3.

3. Proofs of the Results

Using Lemma 3, in particular (9), and taking $\alpha < \gamma$ into account we see that

$$\sum_{n=-\infty}^{\infty} u(n)\, H(z-n) + u(n+i)\, H(\bar{z}-n+i) =: v(z) \tag{14}$$

converges uniformly on every compact subset of $\mathbb{C}$. Thus, this series represents an entire harmonic function v. Moreover, v must be of exponential type $\tau := \pi + \delta < 2\pi$ as we may conclude by estimating the growth of $|H(x+iy)|$ in dependence of y using (13) and well-known growth properties of entire functions of exponential type (see, e.g., [2, Sect. 6.2]).

To the given harmonic function u there exists an entire function f of exponential type σ such that $u = \operatorname{Re} f$. Consider now the entire functions g and h defined by

$$g(z) := \tfrac{1}{2}(f(z) + \overline{f(\bar{z})}) \tag{15}$$

and

$$h(z) := \tfrac{1}{2}(f(z+i) + \overline{f(\bar{z}+i)}). \tag{16}$$

They are both of exponential type $\sigma < \pi$ and

$$g(x) \equiv u(x), \qquad h(x) \equiv u(x+i) \tag{17}$$

for real x. Furthermore, for $z = x$ and $z = x + i$ the series (14) reduces to the one in (7) for g and h, respectively. Hence by Lemma 2

$$u(x) \equiv v(x) \qquad \text{and} \qquad u(x+i) \equiv v(x+i)$$

for real x.

Now let $w := u - v$, which is a real-valued entire harmonic function of exponential type τ vanishing identically on $\mathbb{R}$ and $i + \mathbb{R}$. Again there exists an entire function F of exponential type τ such that $w = \operatorname{Re} F$. Next, from (15)–(17) with f and u replaced by F and w, respectively, we deduce

$$F(z) \equiv -\overline{F(\bar{z})} \qquad \text{and} \qquad F(z+i) \equiv -\overline{F(\bar{z}+i)},$$

implying

$$F(z+2i) \equiv F(z)$$

for all $z \in \mathbb{C}$, i.e., F has period $2i$. As is well-known [2, Theorem 6.10.1] F must then be of the form

$$F(z) = \sum_{j=-N}^{N} A_j \exp(j\pi z).$$

Moreover, $A_j = 0$ if $|j| \geqslant 2$, since $\tau < 2\pi$. Hence

$$\begin{aligned} w(x+iy) &= a + (b_1 e^{\pi x} + b_2 e^{-\pi x}) \cos \pi y \\ &\quad + (c_1 e^{\pi x} + c_2 e^{-\pi x}) \sin \pi y \end{aligned} \tag{18}$$

for some real constants a, b_1, b_2, c_1, and c_2. Since $w \equiv 0$ on $\mathbb{R}$ and $i + \mathbb{R}$ we obtain $a = b_1 = b_2 = 0$.

Finally, consider (14) with $z = x + i/2$, where $x > 0$. Splitting the summation into

$$\sum_{n=-\infty}^{\infty} \cdots = \sum_{n=-\infty}^{0} \cdots + \sum_{n=1}^{[2x]} \cdots + \sum_{n=[2x]+1}^{\infty} \cdots,$$

we find, already by crude estimates, using Lemma 3 that

$$|v(x+i/2)| = O(|x| \exp(c\,|x|/|\log x|^{\alpha})) \qquad \text{as} \quad x \to +\infty,$$

where c is a positive constant. It is similarly seen that the same asymptotic behaviour holds for $x \to -\infty$. Hence

$$\lim_{x\to\infty} e^{-\pi x} v(x+i/2) = 0, \qquad \lim_{x\to-\infty} e^{\pi x} v(x+i/2) = 0. \tag{19}$$

Since u is given to be of exponential type less than π we also have

$$\lim_{x\to\infty} e^{-\pi x} u(x+i/2) = 0, \qquad \lim_{x\to-\infty} e^{\pi x} u(x+i/2) = 0. \tag{20}$$

Thus (19) and (20) show that in (18) the constants c_1 and c_2 must be zero, i.e., $u(z) \equiv v(z)$. This completes the proof of the theorem.

In the situation of the corollary our theorem applies with any $\gamma > 1$. Putting

$$L(z) := \sum_{n=-\infty}^{\infty} |H(z-n)| + |H(\bar{z}-n+i)| \tag{21}$$

we readily see with the help of (9) that the function L is bounded on every compact subset of $\mathbb{C}$. From (5) we deduce

$$|u(z)| \leqslant K \cdot L(z).$$

But L has period 1, i.e.,

$$L(z+1) \equiv L(z)$$

as (21) shows. Hence, if R denotes the rectangular region

$$R := \{z : 0 \leqslant \operatorname{Re} z \leqslant 1, |\operatorname{Im} z| \leqslant y_0\},$$

we see that

$$\sup_{|\operatorname{Im} z| \leqslant y_0} L(z) = \sup_{z \in R} L(z) =: M < +\infty$$

which completes the proof of the corollary.

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